COHOMOLOGICAL AMPLITUDE FOR CONSTRUCTIBLE SHEAVES ON MODULI SPACES OF CURVES

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ABSTRACT. We prove that the cohomology of a constructible abelian sheaf $\mathcal{F}$ on the moduli stack $\mathcal{M}_{g,n}(\mathbb{C})$ (for the Euclidean topology) is zero in degree $\geq g + \dim \operatorname{supp}(\mathcal{F})$. This implies Harer's bound for the homotopy type of $\mathcal{M}_{g,n}(\mathbb{C})$ and the bound of Diaz on the maximal dimension of a complete subvariety of $\mathcal{M}_{g,n}(\mathbb{C})$. We also obtain this type of bound for any open subset of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}(\mathbb{C})$ that is a union of strata. These imply conjectures of Roth and Vakil. Furthermore, our proof provides a template for obtaining similar bounds on the cohomological dimension of $\mathcal{M}_{g,n}(\mathbb{C})$ for quasi-coherent sheaves.

INTRODUCTION

Let X be a complex variety. We recall that an abelian sheaf $\mathcal{F}$ on X is said to be *constructible* if there exists a locally finite partition of X into subsets that are locally closed for the Zariski topology such that the restriction of $\mathcal{F}$ to each member of that partition is locally constant for the Euclidean topology (its stalks may be arbitrary). This notion is what we end up with if we are looking for a category of sheaves that contains the locally constant sheaves over smooth varieties and is stable under all 'natural' functors. To make this somewhat more concrete, let $\mathcal{F}$ be a constructible sheaf on the variety X. If W denotes its support, then there is a closed lower dimensional subvariety $Y \subset W$ such $W - Y$ is smooth and $\mathcal{F}|W - Y$ is a locally constant. If we denote by $j : W - Y \subset X$ and $i : Y \subset X$ the inclusions, then we have on X the short exact sequence of constructible sheaves

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

and this quickly leads to the observation that $\mathcal{F}$ admits a filtration $0 = \mathcal{F}_{d+1} \subset \mathcal{F}_d \subset \cdots \subset \mathcal{F}_0 = \mathcal{F}$ of which each successive quotient $\mathcal{F}_k/\mathcal{F}_{k-1}$ is of the form $j_! \mathbb{F}$, where $j : S \subset X$ is a smooth subvariety of dimension k and $\mathbb{F}$ is a locally constant on S. (We recall that $j_! \mathbb{F}$ is its extension to X by zero and that $H^\bullet(X, j_! \mathbb{F})$ can be understood as the cohomology of $\mathbb{F}$ with support that is closed in X for the Euclidean topology or more intuitively, as the cohomology of the pair $(\bar{S}, \partial S)$ with values in $\mathbb{F}$.)

The *cohomological dimension for constructible sheaves* or briefer, the *constructible cohomological dimension*, $\operatorname{ccd}(X)$, of a variety X is the smallest integer d with the property that the cohomology (for the Euclidean topology)

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of every (constructible) sheaf on X vanishes in degree $> d$. It is well-known that $\operatorname{ccd}(X) \leq 2 \dim X$ and that $\operatorname{ccd}(X) \leq \dim X$ when X is affine. But for our purposes the following stronger notion is more useful.

Definition 0.1. The *cohomological excess* of a nonempty variety X, denoted $\operatorname{ce}(X)$, is the maximum of the integers $\operatorname{ccd}(W) - \dim W$, where W runs over all the Zariski closed subsets $W \subset X$.

In view of the above, we see that this maximum is already attained by a sheaf gotten by extension by zero of some locally constant sheaf on a smooth irreducible subvariety and that $0 \leq \operatorname{ce}(X) \leq \dim X$ with the bounds attained for X is affine resp. X complete. So this notion behaves very much like the cohomological dimension $\operatorname{cd}(X)$ of X for quasi-coherent $\mathcal{O}_X$-modules. Indeed, our arguments will also work for $\operatorname{cd}(X)$ in the sense that if in the above definition of $\operatorname{ce}(X)$ is replaced by $\operatorname{cd}(X)$, then they remain valid with only minor modifications. We have however chosen to focus here on the more delicate notion of cohomological excess—the quasi-coherent cohomological dimension is somewhat easier to deal with since it only involves the Zariski topology. Here is our main result:

Theorem 0.2. *The cohomological excess of the universal curve $\mathcal{C}_g$ of positive genus g is $\leq g-1$. In particular, the cohomological dimension of $\mathcal{C}_g$ for constructible sheaves is at most $\leq (3g-2)+g-1 = 4g-3$.*

A variant of a theorem of M. Artin states that if there exists an affine morphism $X \to Y$, then $\operatorname{ce}(X) \leq \operatorname{ce}(Y)$. Since the forgetful morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,1} = \mathcal{C}_g$ is affine, we immediately get:

Corollary 0.3. *The cohomological excess of the moduli space of pointed curves $\mathcal{M}_{g,n}$ with g and n positive is $\leq g-1$. In particular, the cohomological dimension of $\mathcal{M}_{g,n}$ for constructible sheaves is $\leq (3g-3+n)+g-1 = 4g-4+n$.*

We shall show that if $X \to Y$ is proper surjective with constant fiber dimension n, then $\operatorname{ce}(X) = \operatorname{ce}(Y) + n$. Since the forgetful morphism $\mathcal{C}_g \to \mathcal{M}_g$ is of that type with $n = 1$, we also find:

Corollary 0.4. *The cohomological excess of the moduli space of curves $\mathcal{M}_g$ ($g \geq 2$) is $\leq g-2$. In particular, the cohomological dimension of $\mathcal{M}_{g,n}$ for constructible sheaves is $\leq (3g-3)+g-2 = 4g-5$.*

A version of the Hurewicz theorem asserts that if a space Z has the homotopy type of a connected CW complex and for some integer n, the singular cohomology of every locally constant sheaf on Z vanishes in degree $> n$, then Z has the homotopy type of a CW complex of dimension at most $\max\{3, n\}$ [10]. Here it is crucial that we do not insist that the stalks be finitely generated. So a complex variety Z has the homotopy type of a CW

complex of dimension at most $\mathrm{ce}(Z)+\dim Z$, except perhaps when this sum is smaller than three[1]. We thus recover a theorem of Harer [6]:

Corollary 0.5 (Harer). *A torsion free subgroup of the mapping class group of a n-pointed closed oriented surface of genus $g \geq 1$ has cohomological dimension $\leq 4g-4+n$ and in case $n=0$, $g>1$, this is even $\leq 4g-5$.*

The omitted case of genus zero with $n \geq 3$ is easily taken care of, for $\mathcal{M}_{0,n}$ is affine of dimension $n-3$ and so $\mathrm{ce}(\mathcal{M}_{0,n})=0$. Hence $\mathcal{M}_{0,n}$ and any torsion free subgroup of the associated mapping class group have cohomological dimension at most $n-3$.

The bounds in question are known to be sharp, for if Γ is a torsion free subgroup of finite index of the mapping class group in genus g with n punctures, then according to Harer [6], Γ is duality group (in the sense of Bieri-Eckmann) of the maximal dimension that Harer's theorem allows. This implies that there exists a Γ-module A that has nontrivial cohomology in that degree. The Γ-module A defines an locally constant sheaf on a finite orbifold cover of $\mathcal{M}_{g,n}$ whose direct image on $\mathcal{M}_{g,n}$ has the same cohomology as that of A.

For a subvariety Y of a variety X we clearly have $\mathrm{ce}(Y) \leq \mathrm{ce}(X)$. If Y is complete of dimension d, then $H^{2d}(Y,\mathbb{Q}) \neq 0$ and so $\mathrm{ce}(X) \geq \mathrm{ce}(Y) \geq d$. Hence our main result also implies the famous theorem of Diaz which states that any complete subvariety of $\mathcal{M}_g$ has dimension at most $g-2$.

We can push this a bit further in the spirit of Roth and Vakil [11]. Recall that the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ has a normal crossing divisor as boundary, at least as a stack or after passing to a finite Galois cover. Since there are a number of interesting open parts of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ that are union of strata (where the stratification is the obvious one), it is of interest to know how much a given stratum of $\overline{\mathcal{M}}_{g,n}$ can contribute to the cohomological excess of such an open union. A stratum S parametrizes stable pointed curves of a fixed topological type and hence the normalizations of these curves are also of a fixed topological type. Let us denote by $r(S)$ the number of the genus zero components therein and if $U \subset \overline{\mathcal{M}}_{g,n}$ is a union of strata, write $r(U)$ for the maximal value that r takes on a stratum in U. We shall see that the following proposition follows in a relatively straightforward manner from Corollary 0.3.

Theorem 0.6. *An open union U of strata of $\overline{\mathcal{M}}_{g,n}$ has the property that $\mathrm{ce}(U) \leq g-1+r(U)$.*

So any complete subvariety of such an open subset U has dimension $\leq g-1+r(U)$ and the homotopy type of U is a finite CW complex of dimension $\leq 4g-4+n+r(U)$. This recovers the theorems of Diaz and

[1]As the case $\dim Z \leq 1$ presents no issue, the only case left open is that of a complex surface Z with $\mathrm{ce}(Z)=0$ in which case we cannot rule out the possibility that its homotopy type requires 3-cells.

Mondello [9]. Among the interesting examples alluded to are the loci $\mathcal{M}_{g,n}^{\mathrm{irr}}$, $\mathcal{M}_{g,n}^{\mathrm{ct}}$, $\mathcal{M}_{g,n}^{\mathrm{rt}}$ parametrizing nodal pointed curves which respectively are irreducible, have compact Jacobian, have a smooth genus g curve as an irreducible component (where in this case we assume $g > 0$). The value of r on these loci are respectively 1, $g+n-2$ and $n-1$ and a stratum yielding this bound is one which parametrizes irreducible curves whose normalization is rational, a pointed curve with whose irreducible components consist of g elliptic curves and $g+n-2$ smooth rational curves (the dual intersection graph is a trivalent tree with the elliptic curves sitting at the ends), a trivalent rooted tree of rational curves ($n-1$ in number) planted on a smooth curve of positive genus g.

The present paper has its origin in our attempts to prove that $\mathcal{M}_g$ is covered by $g-1$ open affines (from which all our results would follow). This question is still open, although there has been recent progress by Fontanari and Pascolutti who have shown [5] that this is so for $g \leq 5$.

We collect in Section 1 a number of basic properties of the notion of cohomological excess. In Section 2 we construct families of branched coverings of the affine line with fixed branching behavior near infinity.These families are minimal in a sense; this implies that we allow singularities which are really bad in that they can swallow a great deal "of singular behavior." This is reflected by the fact that these families are not flat over their base and so are not families in the conventional sense. In Section 3 we do a similar construction for curves equipped with pencils. These two constructions are at the heart of the proof of our main theorem.

In this paper schemes are separated and of finite type over $\mathbb{C}$ and carry the Zariski topology. But in order to optimize our results (so that we can for instance invoke the generalized Hurewicz theorem), we use the Euclidean topology when we deal with sheaves and their cohomology, as we would in a complex-analytic setting. For example, the stalk of a higher direct image of a constructible sheaf is understood in terms of the Euclidean topology. We sometimes pass from one to the other via the étale site. Still, since our arguments are essentially algebraic, we believe that the proofs can be modified to yield the corresponding results for constructible ℓ-adic sheaves on $\mathcal{M}_{g,n}(k)$, where k is an algebraically closed field whose characteristic does not divide ℓ and where we work solely with the étale topology. (When k is of characteristic zero, this is in fact the case.)

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especially Gabriele Mondello, whose comments led me to improve the exposition. I am indebted to the late Frans Clauwens for pointing me long ago to the generalized Hurewicz theorem.

Conventions. For a sheaf $\mathcal{F}$ (on a scheme), we denote by $\dim \mathcal{F}$ the dimension of the support of $\mathcal{F}$.

By a *stratification (of depth $\leq n$)* of a variety X we mean a nested sequence $X^\bullet = (X = X^0 \supset X^1 \supset \cdots \supset X^{n+1} = \emptyset)$ of closed subsets; a *stratum* is then a connected component of some successive difference $X^k - X^{k+1}$. We say that a stratification is *affine* when its strata are.

If $c \in \mathbb{R}$, then c_+ is short for $\max\{c, 0\}$ and we use the same notation for a real valued function.

1. Cohomological excess

We collect in this section some general properties of this notion, but we advise the reader to consult this section only when the need arises. For basic facts about cohomology of constructible sheaves we refer to [7] and SGA 4. As we mentioned in the introduction, we do not adopt the algebraic definition based on the étale topology, but one that is transcendental in nature. It would therefore seem more logical to choose for the complex-analytic setting (the notion of a constructible sheaf in this context is then the obvious one), but this has other drawbacks, one being that the direct image of such a sheaf under a complex-analytic morphism need not be of the same type, unless the morphism is proper. This hybrid approach perhaps also accounts for our difficulty in finding references for the results we need.

The title of this paper is explained by the observation (which is however not used in what follows) that our notion of cohomological excess is expressible in terms of the middle perversity as *cohomological amplitude*. To be precise, for a scheme X, denote by $\mathcal{D}^b_c(X)$ the derived category of constructible sheaves with bounded cohomology. Following [4], the middle perversity defines a *t-structure* on $\mathcal{D}^b_c(X)$ for which $\mathcal{D}^b_c(X)^{\leq r}$ is represented by the complexes $\mathcal{K}$ on X with the property that each $\mathcal{H}^i(\mathcal{K})$ is constructible, has support of dimension $\leq r-i$, and is zero in all but a finite number of degrees. Then $\mathrm{ce}(X) \leq d$ is equivalent to: the map $p_X : X \to o$ to a singleton has the property that p_{X*} sends $\mathcal{D}^b_c(X)^{\leq 0}$ to $\mathcal{D}^b_c(o)^{\leq d}$. (But note that we did not impose any finiteness condition on the stalks of our sheaves, so that in this setting we cannot invoke Verdier duality in a straightforward manner.)

The following theorem is in the étale setting due to M. Artin (for torsion sheaves this is Theorem. 3.1 of Exposé XIV of SGA 4; see also [4] Theorem. 4.1.1).

Proposition 1.1. *Let $f : X \to Y$ be an affine morphism. Then f_* is right exact relative to the t-structure defined above: if $\mathcal{F}$ is a constructible sheaf on X and q an integer ≥ 0, then $q + \dim \mathcal{R}^q f_*\mathcal{F} \leq \dim \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a constructible sheaf on X. Without loss of generality we may assume that X is affine and irreducible, that $\operatorname{supp}\mathcal{F} = X$ and that $f(X)$ is dense in Y. We stratify Y in such a manner that $(\mathcal{F}, f)$ is topologically locally constant in the Euclidean topology along each stratum in the sense that for every stratum S, $R^q f_*\mathcal{F}$ is locally constant along S and its stalk at a closed point of S is computed as the cohomology of $\mathcal{F}$ on the preimage of a small Stein slice $\tilde{S}^\perp$ to S in X. Then $f^{-1}\tilde{S}^\perp$ is Stein and of dimension $\leq \dim X - \dim S$. So $R^q f_*\mathcal{F}|S \neq 0$ implies $q \leq \dim X - \dim S$. Hence $q + \dim R^q f_*\mathcal{F} \leq \dim X = \dim \mathcal{F}$. □

Corollary 1.2. *In this situation* $\mathrm{ce}(X) \leq \mathrm{ce}(Y)$.

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

If $E_2^{p,q} \neq 0$, then $p \leq \dim R^q f_*\mathcal{F} + \mathrm{ce}(Y)$ by definition. If we combine this with Proposition 1.1, we find that $p + q \leq \mathrm{ce}(Y) + \dim \mathcal{F}$. This proves that $\mathrm{ce}(X) \leq \mathrm{ce}(Y)$. □

Here are some more results of this type.

Proposition 1.3. *Let $f : X \to Y$ be a proper surjective morphism of varieties, all of whose fibers have dimension $\leq k$. Then* $\mathrm{ce}(X) \leq \mathrm{ce}(Y) + k$. *Equality holds if all the fibers of f are nonempty and of the same dimension k. In particular, if f is finite and onto, then* $\mathrm{ce}(X) = \mathrm{ce}(Y)$.

Proof. The inequality $\mathrm{ce}(X) \leq \mathrm{ce}(Y) + k$ is a general fact, see [4], (4.2.4).

To prove the last assertion, choose a constructible sheaf $\mathcal{F}$ on Y which realizes $\mathrm{ce}(Y)$. We may do this in such a manner that $\mathcal{F}$ has irreducible support W, so that if we put $d := \mathrm{ce}(Y) + \dim W$, then $H^d(Y, \mathcal{F}) \neq 0$. The standard orientations define a trace homomorphism (integration along the fiber) $Tr : R^{2k} f_* f^*\mathcal{F} \to \mathcal{F}$. This map is surjective on an open-dense subset of W so that its cokernel $\mathcal{C}$ is supported by a proper subvariety of Y: $\dim \mathcal{C} < \dim W$. This implies that $H^d(Y, \mathcal{C}) = 0$. Its kernel $\mathcal{K}$ is also constructible, and hence has no cohomology in degree $d + 1$. It follows that $Tr : H^d(Y, R^{2k} f_* f^*\mathcal{F}) \to H^d(Y, \mathcal{F})$ is surjective. In particular, $H^d(Y, R^{2k} f_* f^*\mathcal{F}) \neq 0$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* f^*\mathcal{F}) \Rightarrow H^{p+q}(X, f^*\mathcal{F}).$$

Since $R^q f_* f^*\mathcal{F} = 0$ for $q > 2k$, it follows that $H^{p+2k}(X, f^*\mathcal{F})$ maps onto $E_2^{p,2k}$. Hence $H^{d+2k}(X, f^*\mathcal{F}) \neq 0$. Since the support of $f^*\mathcal{F}$ is of dimension $k + \dim W$, it follows that $\mathrm{ce}(X) \geq (d + 2k) - (k + \dim W) = \mathrm{ce}(Y) + k$. □

The last clause of this proposition shows that for the cohomological excess of the moduli spaces of concern here we need not specify whether we regard them as a Deligne-Mumford stack or as a coarse moduli space.

We shall need the following proposition for affine stratifications only, but our convention regarding the use of the Euclidean topology forces us to step outside the algebraic category (an algebraic version appears in [11]).

Proposition 1.4. *Let X be a complex-analytic variety and $X^\bullet = (X = X^0 \supset X^1 \supset \cdots \supset X^{n+1} = \emptyset)$ a complex-analytic Stein stratification of length $\leq n$ (in the sense that each successive difference $X^k - X^{k+1}$ is Stein). Then the cohomological excess of X is at most n in the complex-analytic sense: if $\mathcal{F}$ be a complex-analytically constructible sheaf on X with support W, then $H^q(X, \mathcal{F}) = 0$ for $q > n + \dim W$. Furthermore, if $i_k : X^k - X^{k+1} \subset X$ denotes the inclusion, then $\mathcal{R}^q(i_k^! \mathcal{F}) = 0$ for $q > k$.*

Proof. Upon replacing $X^\bullet$ by its restriction to W, we see that we may assume that $W = X$. For $n = 0$, X is Stein and then this is well-known. We proceed with induction on n. So let $n > 0$. The induction hypothesis applied to $X^\bullet | X - X^n$ implies that $\mathrm{ce}(X - X^n) \leq n - 1$. If $\mathcal{F}$ is a constructible sheaf on X, then we have a long exact cohomology sequence

$$\cdots \to H^k_{X^n}(X, \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X - X^n, \mathcal{F}) \to \cdots$$

Since $H^k(X - X^n, \mathcal{F}) = 0$ for $k \geq n$, it remains to show that $H^k_{X^n}(X, \mathcal{F}) = 0$ for $k > n$. Since X^n is Stein, the Leray spectral sequence for local cohomology at X^n degenerates so that $H^k_{X^n}(X, \mathcal{F}) = H^0(X^n, \mathcal{R}^k i_n^! \mathcal{F})$. Recall that $\mathcal{R}^k i_n^! \mathcal{F}$ is associated to the presheaf which assigns to $U \subset X$ open, $H^k_{U \cap X^n}(U, \mathcal{F})$. If U is Stein, then for $k > 0$, $H^k(U, \mathcal{F}) = 0$ and so the boundary map $H^{k-1}(U \backslash X^n, \mathcal{F}) \to H^k_{U \cap X^n}(U, \mathcal{F})$ is onto. The induction hypothesis applied to $X^\bullet | U \setminus X^n$ implies that $H^{k-1}(U \setminus X^n, \mathcal{F}) = 0$ for $k - 1 > n - 1$ and hence that $H^k_{U \cap X^n}(U, \mathcal{F}) = 0$ for $k > n$. $\square$

We extend the notion of cohomological dimension to certain functors defined on the category of constructible sheaves. If X is a variety, $i : Y \subset X$ a (locally closed) subvariety and $\mathcal{F}$ constructible sheaf on X, then (as agreed) we use for the definition of $\mathcal{R}^q i^! \mathcal{F}$ the Euclidean topology. So an element of its stalk at a is for $q = 0$ represented by an element of $H^0(U, \mathcal{F})$ with support contained in $Y \cap U$ and for $q > 0$ by an element of $H^{q-1}(U \setminus Y, \mathcal{F})$, where U is a *Euclidean* neighborhood of a in X (which we may take to be Stein).

Definition 1.5. Let X be a variety and $i : Y \subset X$ a (locally closed) subvariety. The *cohomological excess of X along Y*, denoted by $\mathrm{ce}(i^!)$, is the maximum of

$$q + \dim(R^q i^! \mathcal{F}) - \dim \mathcal{F},$$

where $\mathcal{F}$ runs over the constructible sheaves on X and q runs over the integers ≥ 0. (So $R^q i^! \mathcal{F} = 0$ whenever $q > \dim(\mathcal{F}) + \mathrm{ce}(i^!)$.)

If $i : Y \subset X$ is closed, then the *cohomological excess with support on Y*, denoted $\mathrm{ce}_Y(X)$, is the smallest integer d for which $H^q_Y(X, \mathcal{F}) = 0$ for all $q > d + \dim \mathcal{F}$.

Remark 1.6. If $f : \tilde{X} \to X$ is finite and open and $i : Y \subset X$ is locally closed with preimage $\tilde{i} : \tilde{Y} \subset \tilde{X}$, then $\mathrm{ce}(\tilde{i}^!) = \mathrm{ce}(i^!)$. This implies that $\mathrm{ce}(i^!)$ only depends on the underlying structure as algebraic spaces. The notion of a constructible sheaf is a priori tied to the structure of a variety and this

is why it is not clear to us whether $\mathrm{ce}(i^!)$ only depends on the underlying analytic structure (let alone of the formal completion of X along Y).

Proposition 1.7. *Let X be a variety and $i : Y \subset X$ a closed subvariety. Then* $\mathrm{ce}(X{-}Y) \le \max\{\mathrm{ce}(X), \mathrm{ce}_Y(X){-}1\}$ *and* $\mathrm{ce}(X) \le \max\{\mathrm{ce}(X{-}Y), \mathrm{ce}_Y(X)\}$. *Moreover,* $\mathrm{ce}_Y(X) \le \mathrm{ce}(Y) + \mathrm{ce}(i^!)$.

Proof. Let $\mathcal{F}$ be a constructible sheaf on X. The first assertion follows from the long exact sequence

$$\dots H^k_Y(X, \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X - Y, \mathcal{F}) \to H^{k+1}_Y(X, \mathcal{F}) \dots$$

For the second assertion we consider the spectral sequence

$$E_2^{p,q} = H^p(Y, R^q i^! \mathcal{F}) \Rightarrow H^{p+q}_Y(X, \mathcal{F}).$$

Notice that $E_2^{p,q} = 0$ if $p > \mathrm{ce}(Y) + \dim R^q i^! \mathcal{F}$. Since $\dim R^q i^! \mathcal{F} \le \dim \mathcal{F} - q + \mathrm{ce}(i^!)$, it follows that $E_2^{p,q} = 0$ if $p + q > \mathrm{ce}(Y) + \mathrm{ce}(i_!) + \dim \mathcal{F}$. Hence $H^k_Y(X, \mathcal{F}) = 0$ for $k > \mathrm{ce}(Y) + \mathrm{ce}(i_!) + \dim \mathcal{F}$ and the proposition follows. $\square$

Corollary 1.8. *Let $X^\bullet = (X = X^0 \supset X^1 \supset \cdots \supset X^{n+1} = \emptyset)$ be a variety endowed with a stratification of depth $\le n$. Denote by $i_k : X^k \subset X$ the inclusion and suppose that for $k \ge 1$, we have $\mathrm{ce}(i_k^!)|X^k - X^{k+1} \le k$ and $\mathrm{ce}(X^k - X^{k+1}) \le (n-1-k)_+$. Then $\mathrm{ce}(X - X^1) \le \max\{\mathrm{ce}(X), n-1\}$ and for $k = 1, \dots, n$, $\mathrm{ce}(X - X^k) \le \max\{\mathrm{ce}(X - X^1), n-1\}$.*

Proof. This indeed follows from a successive application of Proposition 1.7 to the pairs $(X - X^{n+1-k}, X - X^{n-k})$, $k = 0, 1, \dots, n-1$: we find that for any constructible sheaf $\mathcal{F}$ on X and $k \ge n + \dim \mathcal{F}$, the chain of natural maps

$$H^k(X, \mathcal{F}) \twoheadrightarrow H^k(X{-}X^n, \mathcal{F}) \xrightarrow{\cong} H^k(X{-}X^{n-1}, \mathcal{F}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H^k(X{-}X^1, \mathcal{F})$$

is, as the display indicates, a surjection followed by isomorphisms. $\square$

Lemma 1.9. *For varieties X and Y we have* $\mathrm{ce}(X \times Y) \le \mathrm{ce}(X) + \mathrm{ce}(Y)$.

Proof. Let $\mathcal{F}$ be a constructible sheaf on $X \times Y$. We prove with induction on $\dim Y$ that $H^r(X \times Y, \mathcal{F}) = 0$ for $r > \dim \mathcal{F} + \mathrm{ce}(X) + \mathrm{ce}(Y)$. We may assume that $W := \mathrm{supp}(\mathcal{F})$ is irreducible and upon replacing Y by the Zariski closure of $\pi_Y(W)$ we may also assume that $\pi_Y(W)$ is dense in Y (so that Y is irreducible as well). Choose a hypersurface $D \subset Y$ that is locally given by a single equation and has the property that $\mathcal{F}$ is topologically locally trivial over $U := Y - D$. Since the inclusions $D \subset Y$ and $U \subset Y$ are affine, we have $\mathrm{ce}(D) \le \mathrm{ce}(Y)$ and $\mathrm{ce}(U) \le \mathrm{ce}(Y)$. In view of the exact sequence

$$\cdots \to H^r_{X \times D}(X \times Y, \mathcal{F}) \to H^r(X \times Y, \mathcal{F}) \to H^r(X \times U, \mathcal{F}) \to \cdots$$

it suffices to show that the terms neighboring $H^r(X \times Y, \mathcal{F})$ vanish for $r > \dim W + \mathrm{ce}(X) + \mathrm{ce}(Y)$. We treat both terms separately.

Consider the spectral sequence

$$E_2^{p,q} = H^p(Y, \mathcal{R}^q \pi_{Y*} \mathcal{F}) \Rightarrow H^{p+q}(X \times Y, \mathcal{F}).$$

Suppose $E_2^{p,q}$ is nonzero. Then $p \le \dim \mathcal{R}^q \pi_{Y*}\mathcal{F} + \mathrm{ce}(Y)$. Let $y \in U$ be a closed point, and let $i_y : x \in X \mapsto (x,y) \in X \times Y$. The local triviality implies that $(\mathcal{R}^q\pi_{Y*}\mathcal{F})_y = H^q(X, i_y^*\mathcal{F}_y)$. Since we assume this stalk to be nonzero, we must have $q \le \dim i_y^*\mathcal{F} + \mathrm{ce}(X)$. Hence $p+q \le \dim i_y^*\mathcal{F} + \dim \mathcal{R}^q\pi_{Y*}\mathcal{F} + \mathrm{ce}(X) + \mathrm{ce}(Y)$. Since $\dim i_y^*\mathcal{F} + \dim \mathcal{R}^q\pi_{Y*}\mathcal{F} = \dim i_y^*\mathcal{F} + \dim Y = \dim W$, we conclude that $H^r(X \times U, \mathcal{F}) = 0$ for $r > \dim W + \mathrm{ce}(X) + \mathrm{ce}(Y)$.

Next we consider $H^r_{X\times D}(X \times Y, \mathcal{F})$. Denote by $i_D : X \times D \subset X \times Y$ the inclusion. We have a spectral sequence

$$E_2^{p,q} = H^p(X \times D, \mathcal{R}^q i_D^! \mathcal{F}) \Rightarrow H^{p+q}_{X\times D}(X \times Y, \mathcal{F}).$$

For $q > 0$ we have $\mathcal{R}^q i_D^!\mathcal{F} = i_D^*\mathcal{R}^{q-1}j_*j^*\mathcal{F}$, where $j : X \times U \subset X \times Y$. Since j is affine, this can only be nonzero for $q = 1$. Notice that $\mathcal{R}^q i_D^!\mathcal{F}$ has its support contained in $W \cap (X \times D)$. This is a hypersurface in W and so of dimension one less than that of W. If we apply our induction hypothesis to $X \times D$ we find that $H^p(X \times D, \mathcal{R}^q i_D^!\mathcal{F}) \neq 0$ implies $q \le 1$ and $p \le \mathrm{ce}(X) + \mathrm{ce}(D) + \dim(W \cap (X \times D)) \le \mathrm{ce}(X) + \mathrm{ce}(Y) + \dim W - 1$ and so $p + q \le \mathrm{ce}(X) + \mathrm{ce}(Y) + \dim W$. It follows that $H^r_{X\times D}(X \times Y, \mathcal{F}) = 0$ for $r > \mathrm{ce}(X) + \mathrm{ce}(Y) + \dim W$. □

Lemma 1.10. *Let X and Y be affine and let $A \subset X$ and $B \subset Y$ be closed subsets. Then $\mathrm{ce}_{A\times B}(X \times Y) \le \mathrm{ce}_A(X) + \mathrm{ce}_B(Y)$.*

Proof. Without loss of generality we assume X and Y connected.

Assume first that $\mathrm{ce}_A(X)$ and $\mathrm{ce}_B(Y)$ are positive. Since X is affine we have $\mathrm{ce}(X) = 0$. On the other hand, we assumed $\mathrm{ce}_A(X) > 0$. From the exact sequence

$$\cdots \to H^{r-1}(X, \mathcal{F}) \to H^{r-1}(X - A, \mathcal{F}) \to H^r_A(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to \cdots$$

it then follows that $\mathrm{ce}(X - A) = \mathrm{ce}_A(X) - 1$ and likewise $\mathrm{ce}(Y - B) = \mathrm{ce}_B(Y) - 1$. This also shows that $\mathrm{ce}_{A\times B}(X \times Y) \le \mathrm{ce}(X \times Y - A \times B) + 1$. For any constructible sheaf $\mathcal{F}$ on $X \times Y$ we have an exact sequence

$$\begin{aligned}\cdots \to H^{r-1}((X - A) \times (Y - B), \mathcal{F}) \to H^r(X \times Y - A \times B, \mathcal{F}) \to \\ \to H^r(X - A) \times Y, \mathcal{F}) \oplus H^r(X \times (Y - B), \mathcal{F}) \to \cdots\end{aligned}$$

If we combine this with Lemma 1.9, we see that

$$\begin{aligned}&\mathrm{ce}_{A\times B}(X \times Y) \le \mathrm{ce}(X \times Y - A \times B) + 1 \le \\ &\quad \max\{\mathrm{ce}(X - A) + \mathrm{ce}(Y - B) + 2, \mathrm{ce}(X - A) + 1, \mathrm{ce}(Y - B) + 1\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le \mathrm{ce}_A(X) + \mathrm{ce}_B(Y).\end{aligned}$$

If $\mathrm{ce}_A(X) = 0$, then let $\mathcal{F}$ be sheaf on X which is $\mathbb{Z}$ on $X - A$ and zero on A. Since $H^0(X, \mathcal{F}) \to H^0(X - A, \mathcal{F}) = H^0(X - A, \mathbb{Z})$ must be surjective, it follows that $A = \emptyset$ or $A = X$. If still $\mathrm{ce}_B(Y) > 0$, then the above arguments are easily modified for these two cases. Likewise if $\mathrm{ce}_B(Y) = 0$. □

Lemma 1.11. *Let $\mathbb{G}_m$ act properly on the variety E, $f : E \to B$ a formation of its orbit space and let $A \subset B$ be closed. Then $\mathrm{ce}_{E|A}(E) = \mathrm{ce}_A(B)$.*

Proof. Since f is affine, it is enough to show that $\mathrm{ce}_{E|A}(E) \geq \mathrm{ce}_A(B)$. Let $\mathcal{F}$ be a constructible sheaf on B with $H^q_A(B,\mathcal{F}) \neq 0$ for $q = \mathrm{ce}_A(B) + \dim \mathcal{F}$. The Leray spectral sequence $E_2^{p,q} = H^p_A(B, R^q f_* f^*\mathcal{F}) \Rightarrow H^{p+q}_{E|A}(E, f^*\mathcal{F})$ reduces tot the Gysin sequence, for we have $E_2^{p,1} \cong E_2^{p,0} = H^p_A(B,\mathcal{F})$ and $E_2^{p,q} = 0$ for $q \neq 0, 1$. In particular, we have a surjection $H^{d+1}_{E|A}(E, f^*\mathcal{F}) \to E_2^{d,1} \cong H^d_A(B,\mathcal{F})$ and so $H^{d+1}_{E|A}(E, f^*\mathcal{F}) \neq 0$. □

The content of the following lemma resides in the fact that our sheaf cohomology uses the Euclidean topology.

Lemma 1.12. *Let X be affine and $i : Y \subset X$ a closed subset. Assume X admits a $\mathbb{G}_m$-action with a unique fixed point o which is good in the sense that it extends to a semigroup action of $(\mathbb{C}, \times)$. Assume that Y is $\mathbb{G}_m$-invariant and $o \in Y$. Then for any variety U, we have $\mathrm{ce}(o_U^* i_U^!) \leq \mathrm{ce}_Y(X)$. (Here $i_U : Y_U \subset X_U$ and $o_U : U \to Y_U$ are the obvious maps.)*

Proof. Our assumption means that $X = \mathrm{Spec}(A_\bullet)$ of a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra $A_\bullet$ with $A_0 = \mathbb{C}$. Then $\mathbb{G}_m$ acts properly on $X^\circ := X - \{o\}$ and its orbit space can be identified with $\mathbb{P}(X) := \mathrm{Proj}(A_\bullet)$. Then $\mathrm{ce}(\mathbb{P}(X)) = \mathrm{ce}(X)$ by Lemma 1.11. Put $\tilde{X} := X \times^{\mathbb{G}_m} \mathbb{C}$. We have an obvious embedding $k : \mathbb{P}(X) \subset \tilde{X}$ and a natural projection $\pi : \tilde{X} \to X$ that can be understood as a weighted homogeneous blowup which has the image of k as exceptional set. Denote by $j : X^\circ \subset \tilde{X}$ the inclusion and by $\pi_U : \mathbb{P}(X)_U \to U$ the projection. Then for a constructible sheaf $\mathcal{F}$ on $X^\circ \times U$, we have that for $q > 0$, $o_U^* R^q i_U^! (j_{U*}\mathcal{F}) = R^{q-1}\pi_{U*}(k_U^* j_{U*}\mathcal{F})$. The sheaf $k_U^* j_{U*}\mathcal{F}$ is constructible on $\mathbb{P}(X)_U$ and so $R^{q-1}\pi_{U*}(k_U^* j_{U*}\mathcal{F})$ is zero for $q > \dim \mathcal{F} + \mathrm{ce}_Y(X)$ by Lemma 1.9. □

2. Branched covers of the projective line

Pseudo-moduli spaces of Riemann covers. Let be given an integer $g \geq 0$, a finite nonempty set I and a map $\boldsymbol{d} : i \in I \mapsto d_i \in \mathbb{Z}_{>0}$ such that $|\boldsymbol{d}| := \sum_i d_i > 1$. Given a scheme S, then a *smooth Riemann covering over S of branch type* $(g, \boldsymbol{d})$ is a pair

$$(f : \mathcal{C} \to \mathbb{P}^1_S, \boldsymbol{p} = (p_i : S \to \mathcal{C}/S)_{i \in I}),$$

where f is a finite morphism such that $\mathcal{C}/S$ is a smooth projective curve of genus g and $\boldsymbol{p}$ is a trivialization of the divisor $f^*\infty_S$, by which we mean a system of pairwise disjoint sections of $\mathcal{C}/S$ such that $f^*(\infty_S) = \sum_{i\in I} d_i(p_i)$ as divisors. A morphism between two such coverings of given branch type $(f' : \mathcal{C}' \to \mathbb{P}^1_{S'}, \boldsymbol{p}') \to (f : \mathcal{C} \to \mathbb{P}^1_S, \boldsymbol{p})$ consists of a morphism $S' \to S$, a fiberwise translation $\mathbb{P}^1_{S'} \to \mathbb{P}^1_S$ (i.e., given by $[t:1] \mapsto [t+a:1]$, where a is some regular function on S') and a morphism $\mathcal{C}' \to \mathcal{C}$ which sends p_i' to p_i which together make up cartesian diagrams.

A family as above defines a discriminant divisor Δ_f in $\mathbb{P}^1_S$. By the Riemann-Hurwitz theorem, the *finite part* of this discriminant Δ_f°, that is, the restriction of Δ_f to $\mathbb{A}^1_S$, has degree $m = m(g, \boldsymbol{d}) := 2g - 2 + |\boldsymbol{d}| + |I|$. This defines a morphism $S \to \mathrm{Sym}^m(\mathbb{A}^1)$. A given family is always isomorphic to one for which the barycenter of Δ_f° is the origin of $\mathbb{A}^1_S$ (this means that the coefficient of z^{m-1} in the monic polynomial defining Δ_f° is zero). As this eliminates the ambiguity in the translation, we say that the family is *normalized*. The corresponding subspace $\mathrm{Sym}^m_0(\mathbb{A}^1)$ of $\mathrm{Sym}^m(\mathbb{A}^1)$ may be identified with the space of monic polynomials of the form $z^m + a_2 z^{m-2} + \cdots + a_m$ and so is isomorphic to $\mathbb{A}^{m-1}$.

Notice that a single normalized member $(C \to \mathbb{P}^1, \boldsymbol{p})$ has finite automorphism group so that we have a Deligne-Mumford stack as a universal object

$$(\mathcal{C}_{\mathcal{M}_g(\boldsymbol{d})}/\mathbb{P}^1_{\mathcal{M}_g(\boldsymbol{d})}, \boldsymbol{p}).$$

It comes with a quasi-finite morphism $\Delta^\circ : \mathcal{M}_g(\boldsymbol{d}) \to \mathrm{Sym}^m_0(\mathbb{A}^1)$.

We have an evident action of $\mathbb{G}_m$ on $\mathbb{P}^1$ whose action on the affine coordinate z is given by $t^*(z) = tz$.

Since a genus zero cover of $\mathbb{P}^1$ can be written down in terms of coordinates, it is instructive to have a look at that case first. Put $d := |\boldsymbol{d}|$ and $r := |I|$ and enumerate the elements of I by $1, \dots, r$. An element of $\mathcal{M}_0(\boldsymbol{d})$ can then be represented by a rational function of the form

$$\frac{w^d + a_1 w^{d-1} + \cdots + a_{d_1-1} w^{d-d_1+1} + a_{d_1+1} w^{d-d_1-1} + \cdots + a_d}{w^{d_2}(w - w_3)^{d_3} \cdots (w - w_r)^{d_r}}$$

with $(a_1, \dots, \widehat{a}_{d_1}, \dots a_d)$ arbitrary and $w_3, \dots, w_r$ all nonzero and pairwise distinct. This representation is unique up an action of the group μ_{d_1} of d_1-th roots of unity (acting tautologically on the w coordinate and thereby on the coefficients). Conversely, any such rational function represents an element of $\mathcal{M}_0(\boldsymbol{d})$. Let us denote by M the parameter space of these rational functions. This is an affine space which carries a family and the morphism $M \to \mathcal{M}_0(\boldsymbol{d})$ defined above is a μ_{d_1}-covering. In particular, $\mathcal{M}_0(\boldsymbol{d})$ is affine. But the morphism $\Delta^\circ : M \to \mathrm{Sym}^{d+r-2}_0(\mathbb{A}^1)$ is in general only quasi-finite. This already occurs when $d = r = 2$: then $M = \mathbb{A}^1$ and $\mathrm{Sym}^{d+r-2}_0(\mathbb{A}^1) \cong \mathbb{A}^1$. The rational function $(w^2 + a)/w$ $(a \neq 0)$ has finite discriminant $\{\pm 2\sqrt{a}\}$ and so we miss the case of a point with multiplicity 2. Indeed, if we let $a \to 0$, then the double cover of $\mathbb{P}^1$ degenerates into two lines which meet over 0.

Lemma 2.1. *The stack $\mathcal{M}_g(\boldsymbol{d})$ is irreducible and the evident morphism $\mathcal{M}_g(\boldsymbol{d}) \to \mathcal{M}_{g,I}$ is affine (for $g = 0$, $|I| \leq 2$, when $\mathcal{M}_{g,I}$ is not defined, read this as: $\mathcal{M}_g(\boldsymbol{d})$ is affine).*

Proof. According to Kluitmann [8] the branched covers of $\mathbb{P}^1$ of genus g with specified branching data over ∞ and otherwise simple branching at specified points make up a single orbit under the Hurwitz action. This means that the locus in $\mathcal{M}_g(\boldsymbol{d})$ defined by the property that Δ° maps to a reduced divisor is

connected. Since this locus is a smooth stack that is open-dense in $\mathcal{M}_g(\boldsymbol{d})$, it follows that $\mathcal{M}_g(\boldsymbol{d})$ is irreducible.

Assume now $2g-2+|I|>0$ and let $(f:\mathcal{C}\to S,\boldsymbol{p})$ represent an S-valued point of $\mathcal{M}_{g,I}$ with S affine. Consider the coherent $\mathcal{O}_S$-module $\mathcal{V}:=f_*\mathcal{O}_\mathcal{C}(\sum_i d_i(p_i))/\mathcal{O}_S$. Its geometric realization V/S is affine. For every $i\in I$ we have an evaluation homomorphism $\mathcal{V}\to p_i^*\mathcal{O}_\mathcal{C}(d_i(p_i))$ whose target is a line bundle. Its kernel defines a hypersurface $V_i\subset V$ and so $V-\cup_{i\in I}V_i$ is also affine. But $V-\cup_{i\in I}V_i$ may be identified with the pull-back of $\mathcal{M}_g(\boldsymbol{d})\to\mathcal{M}_{g,I}$ over S. □

Denote by $\hat{\mathcal{M}}_g(\boldsymbol{d})\to\mathrm{Sym}_0^m(\mathbb{A}^1)$ the normalization of $\mathrm{Sym}_0^m(\mathbb{A}^1)$ in $\mathcal{M}_g(\boldsymbol{d})$. It is irreducible because $\mathcal{M}_g(\boldsymbol{d})$ is. Since $\mathrm{Sym}_0^m(\mathbb{A}^1)$ is affine, so is $\hat{\mathcal{M}}_g(\boldsymbol{d})$. Denote by $\mathcal{C}_{\hat{\mathcal{M}}_g(\boldsymbol{d})}\to\mathbb{P}^1_{\hat{\mathcal{M}}_g(\boldsymbol{d})}$ the normalization of $\mathbb{P}^1_{\hat{\mathcal{M}}_g(\boldsymbol{d})}$ in $\mathcal{C}_{\mathcal{M}_g(\boldsymbol{d})}$. We thus get a 'family'

$$(\mathcal{C}_{\hat{\mathcal{M}}_g(\boldsymbol{d})}/\mathbb{P}^1_{\hat{\mathcal{M}}_g(\boldsymbol{d})},\boldsymbol{p}).$$

We write family in quotes, because its structure map need not be flat: the arithmetic genus of a member curve may be drop to become less than g. Still for every such curve we have defined (essentially by construction) a finite discriminant of degree m as if it were of arithmetic genus g. This is why for many purposes we may pretend a member curve to be of arithmetic genus g as long as the finite discriminant is part of the data. Since $\hat{\mathcal{M}}_g(\boldsymbol{d})$ is irreducible, the $\mathbb{G}_m$ action is what in the literature is called 'good': it has a unique fixed point $o\in\hat{\mathcal{M}}_g(\boldsymbol{d})$ which lies in the closure of every $\mathbb{G}_m$-orbit. This makes $\hat{\mathcal{M}}_g(\boldsymbol{d})$ a quasi-homogeneous cone with vertex o. The corresponding cover $C_o\to\mathbb{P}^1$ has a unique singular point x_o (the unique fixed point of the $\mathbb{G}_m$-action on $\mathcal{C}_o$) which solely accounts for the finite discriminant (which is 0 with multiplicity m). From the map-germ $C_{x_o}\to\mathbb{A}^1$ we can read off I and $\boldsymbol{d}$: I is the index set of the local branches of C_{x_o} and d_i is the degree of the ith branch over $\mathbb{A}^1$. But we need the number m to assign to C_{x_o} a local genus and a delta invariant:

$$\begin{aligned} g(C_{x_o},m) &:= \tfrac{1}{2}(m-|I|-|\boldsymbol{d}|)+1,\\ \delta(C_{x_o},m) &:= g(C_{x_o},m)+|I|-1=\tfrac{1}{2}(m+|I|-|\boldsymbol{d}|). \end{aligned}$$

When m is understood, we call the map-germ $C_{x_o}\to\mathbb{A}^1$ (and any germ isomorphic to it) a branch germ of type $(g,\boldsymbol{d})$ and we may write $\delta^\psi(C_{x_o})$ for $\delta(C_{x_o},m)$ and refer it as the *pseudo-delta invariant* of C_{x_o}. We also write $\delta(g,I)$ or $\delta(g,\boldsymbol{d})$ for $g-1+|I|$.

The difference $\hat{\mathcal{M}}_g(\boldsymbol{d})-\mathcal{M}_g(\boldsymbol{d})$ is a hypersurface in $\hat{\mathcal{M}}_g(\boldsymbol{d})$, but there is no reason to expect that it can locally be given by a single equation. Coming to terms with its failure to support a Cartier divisor accounts for much of the work involved here.

Kontsevich modification by stable maps. The following construction of Kontsevich (which in the present case is a minor variation of the Knudsen-Deligne-Mumford compactification) is helpful, though not used, in understanding what goes on in the remainder of this section. Given a scheme S, consider systems

$$(f : \mathcal{C}/S \to \mathbb{P}^1_S, \boldsymbol{p} = (p_i)_{i\in I}),$$

where $\mathcal{C}/S$ a connected projective normal crossing curve over $\mathbb{P}^1_S$ and $\boldsymbol{p}$ is a system of pairwise disjoint sections of the smooth part of $\mathcal{C}/S$ such that $f^*(\infty_S) = \sum_i d_i(p_i)$. We impose the customary stability condition: if a connected component C of a f-fiber (over $z_s \in \mathbb{P}^1_S$, say) is positive dimensional and C' denotes the closure of $C_{z_s} - C$ and $I_C := C \cap C'$, then (C, I_C) is Deligne-Mumford stable. Such a curve in a fiber contributes $m_C(z_s)$ to the discriminant of $f_s : C_s \to \mathbb{P}^1_s$, where m_C is computed as follows: if $d_C : I_C \to \mathbb{N}$ assigns to $p \in I_C$ the local degree of $C' \to \mathbb{P}^1$ at p, then $m_C = 2g(C) - 2 + |d_C| - |I_C|$. The coefficient of z_s in $\Delta^\circ(f_s)$ is then obtained by taking the sum over all such curves C over z and the usual contribution of points in $f_s^{-1}z_s$, where C_s is smooth. In other words, if $C_s^h \subset C_s$ denotes the union of the irreducible components of C_s that map onto $\mathbb{P}^1_s$ (so that $f_s|C_s^h$ is finite), then

$$\Delta^\circ(f_s) = \Delta^\circ(f_s|C_s^h) + \sum_C (2g(C) - 2)(z_C),$$

where the sum is over all positive dimensional connected components of fibers of $f_s : C_s \to \mathbb{P}^1_s$. We may normalize the discriminant as before, and then, essentially following Kontsevich, there is a universal such family. We normalize the base of this family and let

$$(\mathcal{K}_{\overline{\mathcal{M}}_g(\boldsymbol{d})} \to \mathbb{P}^1_{\overline{\mathcal{M}}_g(\boldsymbol{d})}, \boldsymbol{p})$$

be the union of its irreducible components that meet $\mathcal{M}_g(\boldsymbol{d})$. The evident $\mathbb{G}_m$-action on $\mathbb{P}^1$ extends to one on this family. The Stein factorization of the projection $\mathcal{K}_{\overline{\mathcal{M}}_g(\boldsymbol{d})} \to \mathbb{P}^1_{\overline{\mathcal{M}}_g(\boldsymbol{d})}$ has as intermediate factor a Riemann covering of branch type $(g, \boldsymbol{d})$. This defines a morphism $\overline{\mathcal{M}}_g(\boldsymbol{d}) \to \hat{\mathcal{M}}_g(\boldsymbol{d})$ that is the identity on $\mathcal{M}_g(\boldsymbol{d})$. We thus get a $\mathbb{G}_m$-equivariant proper morphism that is covered by a morphism $\mathcal{K}_{\overline{\mathcal{M}}_g(\boldsymbol{d})} \to \mathcal{C}_{\hat{\mathcal{M}}_g(\boldsymbol{d})}$.

Notice that if $\tilde{s} \in \overline{\mathcal{M}}_g(\boldsymbol{d})$ lies over $s \in \hat{\mathcal{M}}_g(\boldsymbol{d})$, then the map from the normalization $\hat{K}^h_{\tilde{s}}$ of $K^h_{\tilde{s}}$ to C_s lifts to an isomorphism $\hat{K}^h_{\tilde{s}} \to \hat{C}_s$ of normalizations. The singularities of $K_{\tilde{s}}$ (or C_s for that matter) define a finite subset $I_{\tilde{s}} \subset \hat{K}^h_{\tilde{s}} \cong \hat{C}_s$. This subset is in fact contained the part $\hat{C}^\circ_s \subset \hat{C}_s$ over $\mathbb{A}^1_s$. By allowing now $\tilde{s}$ to vary, but subject to the condition that $\hat{K}^h_{\tilde{s}}$ stays fixed and equal to $\hat{C} := \hat{C}_s$, we obtain a connected subvariety $\tilde{S} \subset \overline{\mathcal{M}}_g(\boldsymbol{d})$. For an open-dense set of $\tilde{s} \in \tilde{S}$, $\hat{C}$ is also the normalization of $K_{\tilde{s}}$ with $K_{\tilde{s}}$ simply being obtained from $\hat{C}$ by identifying the members of a number (k, say) of point pairs in $\hat{C}^\circ$. Then the definition of the Kontsevich moduli space

shows that by assigning to $\tilde{s} \in \tilde{S}$ this unordered k-tuple of pairs (whose image is $I_{\tilde{s}}$), we have defined a morphism $\tilde{S} \to \mathrm{Sym}^k(\mathrm{Sym}^2 \hat{C}^\circ)$ and that this morphism has closed image.

Remark 2.2. It can shown that the morphism $\overline{\mathcal{M}}_g(\boldsymbol{d}) \to \hat{\mathcal{M}}_g(\boldsymbol{d})$ factors through a family which parametrizes curves of that of curves that are *flat* over $\mathbb{P}^1$ (so these curves have no vertical components and still have arithmetic genus g). This family of curves is closely related to one of the compactifications of $\mathcal{M}_{g,I}$ constructed by D. Smyth (Example 1.12 in [12]). The fibers of this morphism will in general have positive dimension.

Local structure of the universal model. Important for what will follow is the observation that the local structure of the pair

$$\Big(\mathcal{C}_{\hat{\mathcal{M}}_g(\boldsymbol{d})} \to \mathbb{P}^1_{\hat{\mathcal{M}}_g(\boldsymbol{d})}, \hat{\mathcal{M}}_g(\boldsymbol{d}) \to \mathrm{Sym}^m_0(\mathbb{A}^1)\Big)$$

is of the same type as the pair itself. Let us make this precise by choosing a closed point $s \in \hat{\mathcal{M}}_g(\boldsymbol{d})$ and a representative $f : C \to \mathbb{P}^1$ of s. For the cover f there is defined a finite discriminant Δ°_f. In fact, at any $x \in C$, singular or not, we have defined the multiplicity m_x of the discriminant of the germ of f at x so that $\sum_{x \in C^\circ} m_x = m$ and a 'degree map' $\boldsymbol{d}_x = (d_{x,i})_{i \in I_x}$ on the set I_x of local branches of (C, x) (which is simply gives the local degree of f on that branch). If x is singular, then we also have defined local genus g_x defined by $g_x := \frac{1}{2}(m_x - |I_x| - |\boldsymbol{d}_x|) + 1$. The analytic germ of the pair above at $x \in C_{\mathrm{sg}}$ is analytically isomorphic to the germ of the pair

$$\Big(\mathcal{C}_{\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)} \to \mathbb{P}^1_{\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)}, \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x) \to \mathrm{Sym}^{m_x}_0(\mathbb{A}^1)\Big)$$

at the $\mathbb{G}_m$-fixed point $o_x \in \mathcal{C}^*_{g_x}(\boldsymbol{d}_x)$ over the origin of $\mathrm{Sym}^{m_x}_0(\mathbb{A}^1)$ times a local factor of dimension $m - m_x$. The germ of the above pair at s is governed by the set C_{sg} of singular points of C in the sense that the smoothings are independent. This is even true in the étale setting:

Lemma 2.3. *We have in fact a natural morphism of étale germs*

$$\Phi_s : \hat{\mathcal{M}}_g(\boldsymbol{d})_s \to \textstyle\prod_{x \in C_{\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x}$$

which lifts to a finite *morphism* $\tilde{\Phi}_s : \hat{\mathcal{M}}_g(\boldsymbol{d})_s \to \mathbb{A}^r_0 \times \prod_{x \in C_{\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x}$ *between étale germs of the same dimension.*

Proof. Each $x \in C_{\mathrm{sg}}$ contributes to the discriminant Δ°_f the term $m_x(f(x))$ so that $\sum_{x \in C_{\mathrm{sg}}} \Delta_x \le \Delta^\circ_f$. Denote by Δ'_f the difference $\Delta^\circ_f - \sum_{x \in C_{\mathrm{sg}}} \Delta_x$ so that $m' := m - \sum_{x \in C_{\mathrm{sg}}} m_x$ is its degree. Consider the morphism

$$\begin{aligned} K : \mathrm{Sym}^{m'}_\nu \mathbb{A}^1 \times \textstyle\prod_{x \in C_{\mathrm{sg}}} \big(\mathbb{G}_a \times \mathrm{Sym}^{m_x}_0 \mathbb{A}^1\big) &\to \mathrm{Sym}^m_0 \mathbb{A}^1, \\ \big(\Delta', (z_x, \Delta_x)_{x \in C_{\mathrm{sg}}}\big) &\mapsto T_{(-\sum_x m_x z_x/m')*}\Delta' + \textstyle\sum_{x \in C_{\mathrm{sg}}} T_{z_x *}\Delta_x, \end{aligned}$$

where T_z denotes the translation over z. This is an étale-local isomorphism at $\tilde{s} := (T_{-1/m'*}\Delta'_f, (f(x), m_x(0))_{x\in C_{\rm sg}})$. We thus obtain a morphism of étale germs:

$$\hat{\mathcal{M}}_g(\boldsymbol{d})_s \cong K^*\hat{\mathcal{M}}_g(\boldsymbol{d})_s \to \mathrm{Sym}^{m'}_{\nu}\,\mathbb{A}^1 \times \textstyle\prod_{x\in C_{\rm sg}}\big(\mathbb{G}_a \times \mathrm{Sym}^{m_x}_0\,\mathbb{A}^1\big).$$

This is a finite morphism. It naturally lifts in the analytic category to an morphism of germs

$$\hat{\mathcal{M}}_g(\boldsymbol{d})_s \to \mathrm{Sym}^{m'}_{\nu}\,\mathbb{A}^1 \times \textstyle\prod_{x\in C_{\rm sg}}\big(\mathbb{G}_a \times \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x).$$

with the property that Φ_s is given by a subproduct. But such a lift then automatically lives in the étale category. □

We denote by $F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ the locus parametrizing curves C with the property that $\sum_{x\in C}\delta^{\psi}(C_x)\geq k$. This is a closed subset of $\hat{\mathcal{M}}_g(\boldsymbol{d})$ and $\mathcal{M}_g(\boldsymbol{d}) = \hat{\mathcal{M}}_g(\boldsymbol{d}) - F^1\hat{\mathcal{M}}_g(\boldsymbol{d})$. Let S be a stratum for this filtration, i.e., a connected component of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$ for some k. It is immediate from the definition that for $s\in S$, the étale germ Φ_s (as defined above) is strict with respect to the filtration $F^\bullet\hat{\mathcal{M}}_g(\boldsymbol{d})_s$ on its source and the convolution of the filtrations $F^\bullet\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{s_x}$ on the range. In particular, the germ of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ at s is the preimage under Φ_s of the product of the $F^{\delta(C_x)}\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{c_x}$, where x runs over all the singular points of a representing curve C_s.

Corollary 2.4. *If for all $s\in S$ and $x\in C_{s,\rm sg}$,*

$$\mathrm{ce}_{F^{\delta(g_x,|I_x|)}\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)}\,\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)\leq \delta(g_x,\boldsymbol{d}_x),$$

then the inclusion $i_S : S\subset \hat{\mathcal{M}}_g(\boldsymbol{d})$ has the property that $\mathrm{ce}(i_S^!)\leq k$.

Proof. In view of Remark 1.6 and Lemma 2.3 it suffices to prove that the inclusion

$$i : \mathbb{A}^r_0 \times \textstyle\prod_{x\in C_{\rm sg}} F^{\delta(g_x,|I_x|)}\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x} \to \mathbb{A}^r_0 \times \prod_{x\in C_{\rm sg}}\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x}$$

has the property that $\mathrm{ce}(i^!)\leq k$ along $\mathbb{A}^r\times((o_x)_{x\in C_{\rm sg}})$. But this follows from Lemmas 1.10 and 1.12, bearing in mind each factor $\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)$ comes with a good $\mathbb{G}_m$-action with fixed point o_x. □

Our definition of pseudo-delta invariant was designed in order to ensure that the usual formula for genus drop still be valid: if $\{C_\alpha\}_{\alpha\in A}$ are the distinct irreducible components of C, then $g-1 = k + \sum_\alpha(g_\alpha - 1)$, where g_α denotes the genus of the normalization $\hat{C}_\alpha$ of C_α. If $I_\alpha\subset I$ denotes the subset picked up by $\hat{C}_\alpha$, then we can also write this as

$$g-1+|I| = k+\textstyle\sum_\alpha(g_\alpha - 1 + |I_\alpha|).$$

We further note that if $(\mathcal{C}/\mathbb{P}^1_B, \boldsymbol{p}_B)$ is family of Riemann covers which defines a morphism $B\to S$, then there exists a simultaneous normalization $\hat{\mathcal{C}}/\mathcal{C}$ of the fibers of $\mathcal{C}/B$ over B.

A property of the strata. We begin with:

Lemma 2.5. *The locus of $c \in F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ for which C_c has k ordinary double points is open and dense in $F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$.*

Proof. In view of the local structure analyzed above, we see that it suffices to treat the case when c is the $\mathbb{G}_m$-fixed point in $\hat{\mathcal{M}}_g(\boldsymbol{d})$ (so that $k = g+|I|-1$). In view of the irreducibility of $\hat{\mathcal{M}}_g(\boldsymbol{d})$ it then suffices to show that $\hat{\mathcal{M}}_g(\boldsymbol{d})$ contains a member with $k = g+|I|-1$ distinct ordinary double points. For this we for take each $i \in I$ a monic polynomial of degree d_i and regard it as defining a morphism $\hat{f}_i : \hat{C}_i = \mathbb{P}^1 \to \mathbb{P}^1$. Let $\hat{f} : \hat{C} \to \mathbb{P}^1$ be their disjoint union. Then choose $g+|I|-1$ distinct fibers of $\hat{f}$ over $\mathbb{A}^1$ and choose in each of these a two-element subset, but make sure that if we identify the points of each of these subsets we get a *connected* nodal cover $f : C \to \mathbb{P}^1$. This curve has arithmetic genus g: we need $|I|-1$ nodes to make it connected so that g nodes remain to give it genus g. Hence $C \to \mathbb{P}^1$ yields an element of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ as desired. □

Remark 2.6. Notice that in the above proof the finite part of the discriminant of $\hat{f}$ is of degree $|\boldsymbol{d}|-|I|$. The introduction of the $k = g+|I|-1$ nodes the adds to this twice an effective divisor of degree $g+|I|-1$ to produce the finite part of the discriminant of f; its degree is indeed $2g-2+|I|+|\boldsymbol{d}|$.

This proof also shows that for $k = g+|I|-1$ the connected components of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$ are in bijective correspondence with the connected graphs Γ with vertex set I and having $g+|I|-1$ edges, where we only allow loops at vertices $i \in I$ with $d_i \geq 2$. To be explicit, if we fix $\hat{f} : \hat{C} \to \mathbb{P}^1$, then the corresponding locus in that connected component is obtained as follows: denote by $\hat{C}^\circ$ the part of $\hat{C}$ that lies over $\mathbb{A}^1$. Then the choice of Γ defines an open subset $X_\Gamma(\hat{f})$ of $\operatorname{Sym}^k \operatorname{Sym}^2_{\mathbb{A}^1} \hat{C}^\circ$. It is easily shown that $X_\Gamma(\hat{f})$ is connected. Notice that the obvious map $X_\Gamma(\hat{f}) \to \operatorname{Sym}^k \mathbb{A}^1$ is quasi-finite. Let $\bar{X}_\Gamma(\hat{f})$ be the closure of $X_\Gamma(\hat{f})$ in $\operatorname{Sym}^k \operatorname{Sym}^2_{\mathbb{A}^1} \hat{C}^\circ$. Its normalization $\hat{X}_\Gamma(\hat{f})$ is also the normalization of $\operatorname{Sym}^k \mathbb{A}^1$ in $X_\Gamma(\hat{f})$. The natural morphism $X_\Gamma(\hat{f}) \to F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ lands in the connected component of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$ defined by Γ. This extends to a finite morphism $\hat{X}_\Gamma(\hat{f}) \to F^k\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$ whose image parametrizes the locus for which the normalization of the branched cover is isomorphic to $\hat{f}$.

The following notion will be useful:

Definition 2.7. The *pseudo-genus* of a complete curve C, denoted $\chi^\psi(C)$, is $\sum_\alpha (g(\hat{C}_\alpha)-1)_+$, where $\{\hat{C}_\alpha\}_\alpha$ is the collection of connected components of the normalization of C.

Its justification comes from the following lemma.

Lemma 2.8. *Let $C \to \mathbb{P}^1$ determine a point of $\mathcal{C}_{\mathcal{M}_g(\boldsymbol{d})} \to \mathbb{P}^1_{\mathcal{M}_g(\boldsymbol{d})}$. If we let r denote the number or rational components of the normalization of C and C_{sg}*

denotes the singular set of C, then

$$\textstyle\sum_{x\in C_{\mathrm{sg}}}\delta^\psi(C_x)+\chi^\psi(C)=g+r-1\le\delta(g,\boldsymbol{d})(=g+|I|-1).$$

Proof. Serre's formula, applied to a nearby smooth fiber of $\mathcal{C}_{\hat{\mathcal{M}}_g(\boldsymbol{d})}/\hat{\mathcal{M}}_g(\boldsymbol{d})$ shows that

$$\textstyle\sum_{x\in C}\delta^\psi(C_x)+\chi^\psi(C)=g-1+r.$$

Since each irreducible component of C contains some p_i, it follows that $r\le|I|$. □

Let S be a connected component of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d})-F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$. After passing to a finite flat base change $B\to S$, we get a family of Riemann coverings $(f:\mathcal{C}_B\to\mathbb{P}^1_B,\boldsymbol{p})$ of type $(g,\boldsymbol{d})$ and which is such that the normalization $\nu:\hat{\mathcal{C}}_B\to\mathcal{C}_B$ defines a smooth morphism $\hat{f}:=f\nu:\hat{\mathcal{C}}_B\to B$ with the property that the connected components of $\hat{\mathcal{C}}_B$ specialize to connected curves. Let A effectively index those connected components. So for $\alpha\in A$ the fibers of the corresponding component $\mathcal{C}_{\alpha,B}\subset\mathcal{C}_B$ are connected with genus g_α, say, and we have an associated subset $I_\alpha\subset I$. If $A_+\subset A$ denotes the set of $\alpha\in A$ for which g_α is positive, then we have a morphism

$$F:B\to\textstyle\prod_{\alpha\in A_+}\mathcal{M}_{g_\alpha,I_\alpha}.$$

Proposition 2.9. *The morphism F is affine. In particular, if $\mathrm{ce}(\mathcal{C}_{g_\alpha,B})\le g_\alpha-1$ for all $\alpha\in A_+$, then $\mathrm{ce}(S)\le\delta(g,\boldsymbol{d})-1-k$.*

Proof. Let $(\hat{\mathcal{C}}_{\check{B}}\to\mathbb{P}^1_{\check{B}},\boldsymbol{p})$ parametrize the smooth Riemann coverings $\hat{C}\to\mathbb{P}^1$ (given up to a translation in $\mathbb{A}^1\subset\mathbb{P}^1$) that are obtained by normalization of the members of B. Clearly, F factors through $\check{B}$. It follows from Lemma 2.1 that the morphism $\check{B}\to\prod_{\alpha\in A_+}\mathcal{M}_{g_\alpha,I_\alpha}$ is affine (the omission of the factors with $g_\alpha=0$ is harmless as the moduli spaces $\mathcal{M}_{0,n}$ are affine). If we select from I_α an element, then we have defined a morphism $\mathcal{M}_{g_\alpha,I_\alpha}\to\mathcal{C}_{g_\alpha}$, which we know, is affine. So it remains to see that the forgetful map $\pi:B\to\check{B}$ is affine.

For this we follow the construction that we carried out in Remark 2.6: in the present case it yields a graph Γ on I as in 2.6, and after possibly a finite base change, a diagram of $\check{B}$-varieties:

$$\mathrm{Sym}^k_{\check{B}}(\mathrm{Sym}^2_{\mathbb{A}^1_{\check{B}}}(\hat{\mathcal{C}}^\circ_{\check{B}}))\supset X_\Gamma(\check{B})\xrightarrow{G}B^\circ\subset B$$

with $X_\Gamma(\check{B})$ open in the left hand side and G a finite morphism to the part B° of B which parametrizes Riemann coverings with ordinary double points only. According to Lemma 2.5, B° is open and dense in B. Notice that $\mathrm{Sym}^k_{\check{B}}(\mathrm{Sym}^2_{\mathbb{A}^1_{\check{B}}}(\hat{\mathcal{C}}^\circ_{\check{B}}))$ is affine over $\check{B}$. So the normalization of the closure of $X_\Gamma(\check{B})$ in $\mathrm{Sym}^k_{\check{B}}(\mathrm{Sym}^2_{\mathbb{A}^1_{\check{B}}}(\check{\mathcal{C}}^\circ))$, denoted $\hat{X}_\Gamma(\check{B})$, is also affine over $\check{B}$. The valuative criterion for properness and Remark 2.6 (which furnishes a local model) show that the $\check{B}$-morphism G extends to a finite surjective morphism $\hat{G}:\hat{X}_\Gamma(\check{B})\to B$. Hence $B\to\check{B}$ is affine. □

3. A PARAMETER SPACE FOR CURVES WITH PENCILS

Let C be a nonsingular complex projective curve of positive genus g, $p \in C$ and P a pencil through $(g+1)(p)$. By Riemann-Roch we have that for a given $p \in C$, $H^0(C, \mathcal{O}_C((g+1)(p)))$ is of dimension ≥ 2 and contains the constants; so the choice of the pencil P amounts to the choice of a point of the projective space of $H^0(C, \mathcal{O}_C((g+1)(p)))/\mathbb{C}$. Then P is the projective space of lines in the dual of the corresponding plane in $H^0(C, \mathcal{O}_C((g+1)(p)))$ and this defines a morphism $f : C \to P$ with p mapping to the point defined by $(g+1)(p)$. Once we identify P with $\mathbb{P}^1$ with $(g+1)(p)$ mapping to ∞ (we call this a *parametrization* of P), then f just becomes a nonconstant rational function f on C with divisor $\geq -(g+1)(p)$. We use ∞ also to denote the point of P that corresponds to $(g+1)(p)$. Denote by r the multiplicity of p as fixed point of P so that the morphism $f : C \to P$ defined by P has degree $g+1-r$. Since $g \geq 1$, this degree must be ≥ 2 and so $0 \leq r \leq g-1$. We define the essential discriminant divisor $\Delta^{\text{ess}}(P)$ of P as the sum of the discriminant divisor $\Delta(f)$ of f and $(2r-g)(\infty)$. A straightforward application of the Riemann-Hurwitz formula shows that $\Delta^{\text{ess}}(P)$ has degree $3g$. Notice that

$$\nu_\infty(\Delta^{\text{ess}}(P)) = \deg(f) - 1 + (2r-g) = (g-r) + (2r-g) = r,$$

So if we write $\Delta^{\text{ess}}(P) = \sum_{i=1}^{3g}(z_i)$ with $z_i \in P$, and if i is the first index for which $z_i \neq \infty$, then $i \leq g$ (for $r \leq g-1$) and we have $\nu_\infty(\Delta^{\text{ess}}(P)) + \nu_{z_i}(\Delta^{\text{ess}}(P)) \leq r + (g-r) \leq g$.

We can associate to the triple (C, p, P) a degree $g+1$ covering of P in the simplest possible way: begin with a disjoint union of C and r copies of P and then identify their points over ∞ as to form a singularity formally isomorphic the union of the $r+1$ coordinates axes $L_0 \cup \cdots \cup L_r$ in $\mathbb{A}^{r+1}$: the resulting curve $C^\sharp$ is connected and maps naturally to P with a singleton (which we denote $p^\sharp$) lying over ∞. The singularity of $C^\sharp$ at $p^\sharp$ is characterized by the property that it has $r+1$ branches and delta-invariant r; it is the unique weakly normal singularity with $r+1$ branches. It may arise as a branched cover as follows: any *connected* union of lines $L'_0 \cup \cdots \cup L'_r$ in $\mathbb{A}^{r+1}$ with L'_i parallel to L_i with r simple nodes (so making up a tree of affine lines) is a deformation of this singularity. If we fix a general linear form $\phi : \mathbb{A}^{r+1} \to \mathbb{A}$, then the restriction of ϕ to $L'_0 \cup \cdots \cup L'_r$ makes this a branched cover whose discriminant is twice the image of the set of nodes (so has degree $2r$); moving these points to $0 \in \mathbb{A}$ (and hence the nodes to $0 \in \mathbb{A}^{r+1}$) yields the degeneration. (This also explains the term $(2r-g)(\infty)$ in the definition of the essential discriminant.)

The corresponding notion over a base B begins with the choice of a pointed genus g curve $(\pi : \mathcal{C} \to B, p : B \to \mathcal{C})$ smooth over B, and a rank two subbundle $\mathcal{E}$ of $\pi_*(\mathcal{O}_\mathcal{C}((g+1)(p)))$ which contains the image of $\mathcal{O}_B$. Then $\mathcal{P} := \check{\mathbb{P}}(\mathcal{E})$ is our relative pencil with $\mathcal{O}_B \subset \mathcal{E}$ defining a section. We then have defined a morphism $\mathcal{C} \to \mathcal{P}$ and the essential discriminant

$\Delta^{\mathrm{ess}}(\mathcal{P})$ in $\mathcal{P}$ as a relative divisor over B. This divisor is effective of degree $3g$. We can describe the relative pencil as a morphism $\mathcal{C}^\sharp \to \mathcal{P}$ of degree $g+1$ as above. We shall call an *(enumerated) trivialization* of the essential discriminant of $(\pi : \mathcal{C} \to B, p : B \to \mathcal{C})$ the given of a $3g$-tuple of sections $\boldsymbol{z} = (z_1, \dots, z_{3g})$ of numbered sections of $\mathcal{P} \to B$ with $\sum_i (z_i) = \Delta^{\mathrm{ess}}(\mathcal{P})$.

Let us first define a parameter space for these trivialized discriminants. Denote by $Z \subset (\mathbb{P}^1)^{3g}$ denote the closed subset defined by $z_1 = \cdots = z_g = \infty$ and put $D_g := (\mathbb{P}^1)^{3g} - Z$. We define a stratification $F^\bullet D_g$ as follows. Let Z_k^i be the closed subset of D_g $(i = 1, \dots, k)$ where simultaneously $z_1 = \cdots = z_i = \infty$ and $\nu_{z_{i+1}}(\Delta_{\boldsymbol{z}}) + \nu_\infty(\Delta_{\boldsymbol{z}}) \geq 1 + k$ (here $\Delta_{\boldsymbol{z}} := \sum_i (z_i)$) and put $F^k D_g := \cup_i Z_k^i$. Clearly $F^k D_g$ is a subvariety of $\mathrm{Sym}^{3g}(\mathbb{P}^1)$ and consists of the $\boldsymbol{z}$ with not all z_i equal ∞ and if i is the first index for which $z_i \neq \infty$ (with necessarily $i \leq g$), then $1 + k \leq \nu_{z_i}(\Delta_{\boldsymbol{z}}) + \nu_\infty(\Delta_{\boldsymbol{z}}) \leq g$.

Notice that when $\boldsymbol{z} \in D_g$, then there exist $1 \leq i < j \leq 3g$ with z_i, z_j distinct and not equal to ∞. So $\mathrm{Aut}(\mathbb{A}^1)$ acts properly on D_g. In fact, the $\mathrm{Aut}(\mathbb{A}^1)$-orbit space $\mathcal{D}_g$ of D_g is a variety: the above property defines an open $\mathrm{Aut}(\mathbb{A}^1)$-invariant subset $U_{ij} \subset D_g$ in which the subvariety $V_{ij} \subset U_{ij}$ defined $z_i = 0$ and $z_j = 1$ represents each $\mathrm{Aut}(\mathbb{A}^1)$-orbit in U_{ij} uniquely. These V_{ij}'s cover $\mathcal{D}_g$ and the coordinate change on $V_{ij} \cap V_{kl}$ is of course given by a morphism $V_{ij} \cap V_{kl} \to \mathrm{Aut}(\mathbb{P}^1)$.

This construction comes with a $\mathbb{P}^1$-bundle $\mathcal{P}_g \to \mathcal{D}_g$ endowed with sections $\infty, z_1, \dots, z_{3g}$. The sum of the z_i's define a divisor Δ_g on $\mathcal{P}_g/\mathcal{D}_g$. The stratification $F^\bullet D_g$ descends to a stratification $F^\bullet \mathcal{D}_g$ of $\mathcal{D}_g$ of depth $g-1$ (for clearly $F^g \mathcal{D}_g = \emptyset$). We claim that this stratification is affine. For if S is a stratum, then the first index i for which $z_i \neq \infty$ and the multiplicities of Δ_g at z_i and ∞ are constant on S. We then can almost trivialize $\mathcal{P}_g|S$ by choosing a coordinate t on $P - \{\infty\}$ such that $t(z_i) = 0$ so that the residual divisor Δ' with support in $\mathbb{A}^1 - \{0\}$ is given by a monic polynomial with constant term 1. This coordinate is unique up to multiplication by a root of 1 of order $\deg(\Delta')$. Thus S is parametrized by an affine space modulo the action of a finite cyclic group and is therefore affine.

It is clear that for a quadruple $(\mathcal{C}/B, p, \mathcal{P}, \boldsymbol{z})$ as above, we have defined an evident morphism $B \to \mathcal{D}_g$ which 'classifies' its trivialized essential discriminant (it lands it the open subset which parametrizes the $3g$-tuples $\boldsymbol{z}$ for which all the multiplicities of $(\infty) + \sum_i (z_i)$ are $\leq g$). We construct a family $(\pi : \mathcal{C} \to \mathcal{B}, p : \mathcal{B} \to \mathcal{C}, \mathcal{P}, \boldsymbol{z})$ with trivialized discriminant that is almost universal as follows. First choose a family of smooth pointed genus g curves $(\pi' : \mathcal{C}_{\mathcal{B}'} \to \mathcal{B}', p' : \mathcal{B}' \to \mathcal{C})$ with smooth base $\mathcal{B}'$ such that the evident morphism $\mathcal{B}' \to \mathcal{C}_g$ is finite and surjective. If we put $\mathcal{B}'' := \mathbb{P}(\pi'_* \mathcal{O}_{\mathcal{C}_{\mathcal{B}'}}((g+1)(p'))/\mathcal{O}_{\mathcal{B}'})$, then $\mathcal{B}'' \to \mathcal{B}'$ is onto and projective and $\mathcal{B}''$ carries a tautological rank two vector bundle $\mathcal{E}$ which contains $\mathcal{O}_{\mathcal{B}''}$. So the associated $\mathbb{P}^1$-bundle $\check{\mathbb{P}}(\mathcal{E}) \to \mathcal{B}''$ comes with a section and we have defined an essential discriminant $\Delta^{\mathrm{ess}}(\check{\mathbb{P}}(\mathcal{E}))$ over $\mathcal{B}''$. We take for $\mathcal{B}$ the $\mathfrak{S}_{3g}$-cover of $\mathcal{B}''$ which enumerates the points of this discriminant. We then

have obtained a family $(\pi : \mathcal{C}_{\mathcal{B}} \to \mathcal{B}, p : \mathcal{B} \to \mathcal{C}_{\mathcal{B}}, \mathcal{P}_{\mathcal{B}}, \boldsymbol{z})$ with the property that the forgetful morphism $\mathcal{C}_{\mathcal{B}} \to \mathcal{C}_g$ is proper and surjective and for which the induced morphism $\mathcal{B} \to \mathcal{D}_g$ is quasi-finite with open-dense image. In particular, $\mathrm{ce}(\mathcal{C}_g) \le \mathrm{ce}(\mathcal{B})$ and so our main theorem will follow if we prove that $\mathrm{ce}(\mathcal{B}) \le g-1$. To this end, we prefer to interpret the family above as one of degree $(g+1)$-coverings of a smooth rational curve:

$$(\mathcal{C}^\sharp_{\mathcal{B}} \to \mathcal{P}_{\mathcal{B}} \to \mathcal{B}, p^\sharp : \mathcal{B} \to \mathcal{C}^\sharp_{\mathcal{B}}, \boldsymbol{z}).$$

Our first goal is to define a partial completion $\bar{\mathcal{B}}$ of $\mathcal{B}$ with $\mathrm{ce}(\bar{\mathcal{B}}) \le g-1$: we let $\bar{\mathcal{B}}$ parametrize systems $(C^\sharp \to P, p, \boldsymbol{z})$ as before, but we now allow the degree $(g+1)$ morphism $C^\sharp \to P$ defined by the pencil to have singularities over $P - \{\infty, z_i\}$ of branch type. This is done as follows. Let $\hat{\mathcal{B}}$ be the normalization of $\mathcal{D}_g$ in $\mathcal{B}$, denote by $\mathcal{P}_{\hat{\mathcal{B}}}$ the pull-back of $\mathcal{P}_g$ under $\hat{\mathcal{B}} \to \mathcal{D}_g$ and let $\mathcal{C}^\sharp_{\hat{\mathcal{B}}}$ be the normalization of $\mathcal{P}_{\hat{\mathcal{B}}}$ in $\mathcal{C}^\sharp_{\mathcal{B}}$. The affine stratification $F^\bullet\mathcal{D}_g$ pulls back to an affine stratification $F^\bullet\hat{\mathcal{B}}$ of $\hat{\mathcal{B}}$. We then take for $\bar{\mathcal{B}} \subset \hat{\mathcal{B}}$ the subset which parametrizes systems of the form $(C^\sharp \to P, p^\sharp, \boldsymbol{z})$ with the property that

(i) $p^\sharp$ is a weakly normal singularity of $C^\sharp$,
(ii) there is a unique connected component of $C^\sharp - \{p^\sharp\}$ which maps not isomorphically to $P - \{\infty\}$ and
(iii) $C^\sharp$ has no singular point over the first point $z_i \neq \infty$ defined by the stratification.

This is an open union of strata of $F^\bullet\hat{\mathcal{B}}$ and hence comes with an affine stratification of depth $g-1$. In particular, $\mathrm{ce}(\bar{\mathcal{B}}) \le g-1$. The $\mathfrak{S}_{3g}$-action on $\mathcal{B}$ extends to $\hat{\mathcal{B}}$, but note that $\bar{\mathcal{B}}$ is not $\mathfrak{S}_{3g}$-invariant. We put $\partial\mathcal{B} := \bar{\mathcal{B}} - \mathcal{B}$ and denote by $F^\bullet\partial\mathcal{B}$ the restriction of $F^\bullet\bar{\mathcal{B}}$ to $\partial\mathcal{B}$. Let S be a connected component of $F^k\partial\mathcal{B} - F^{k-1}\partial\mathcal{B}$ with $k \ge 1$. The local structure of the embedding $i : S \subset \bar{\mathcal{B}}$ is to some extent familiar: if $c \in S$, then for every $x \in C_{c,\mathrm{sg}}$, $x \neq p$, the map $f_c : C_c \to P_c$ has a branch type $(g_x, \boldsymbol{d}_x)$. The argument used in Lemma 2.3 shows that we have a morphism of a étale germs

$$\bar{\mathcal{B}}_c \to \textstyle\prod_{x \in C_{c,\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x},$$

where $o_x \in \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)$ is the fixed point of the $\mathbb{G}_m$-action, with the property that it lifts to a finite morphism $\bar{\mathcal{B}}_c \to \mathbb{A}^r_0 \times \prod_{x \in C_{c,\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x}$ between étale germs of the same dimension and is such that $\partial\mathcal{B}_c$ is the preimage of $F^1\big(\prod_{x \in C_{c,\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)\big)$. This will enable us to estimate the cohomological excess of $\bar{\mathcal{B}}$ along $\partial\mathcal{B}$ in terms of that of $\hat{\mathcal{M}}_h(\boldsymbol{d})$ along $F^1\hat{\mathcal{M}}_h(\boldsymbol{d})$ for $h \le g$:

Corollary 3.1. *Suppose we know that for every pair $(h, \boldsymbol{d})$ with $h \le g$ and $|\boldsymbol{d}| \le g+1$, we have $\mathrm{ce}_{F^{\delta(h,\boldsymbol{d})}\hat{\mathcal{M}}_h(\boldsymbol{d})} \hat{\mathcal{M}}_h(\boldsymbol{d}) \le \delta(h, \boldsymbol{d})$. Then $\mathrm{ce}(\mathcal{C}_g) \le g-1$.*

Proof. Since a stratum S of $F^\bullet\partial\mathcal{B}$ is affine, we have $\mathrm{ce}(S) = 0$ and in view of Corollary 1.8 it then suffices to show that $\mathrm{ce}(i^!_S) \le g$. In view of the

preceding discussion this in turn will follow if we show that for every $c \in S$ as above,

$$i : \mathbb{A}_0^r \times \prod_{x \in C_{c,\mathrm{sg}}} F^{\delta(g_x, |I_x|)} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x} \to \mathbb{A}_0^r \times \prod_{x \in C_{c,\mathrm{sg}}} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)_{o_x}$$

has that property at $(0, (o_x)_{x \in C_{c,\mathrm{sg}}})$. To see this, note that every factor we have $|\boldsymbol{d}_x| \leq g+1$ and that $\sum_{x \in C_{c,\mathrm{sg}}} \delta(g_x, \boldsymbol{d}_x) \leq g$. Our assumption implies then that $\mathrm{ce}_{F^{\delta(g_x,\boldsymbol{d}_x)} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x) \leq \delta(g_x, \boldsymbol{d}_x)$.

First note that for every $c \in S$ and every $x \in C_{c,\mathrm{sg}}$ we have $|\boldsymbol{d}_x| \leq g+1$ and $\sum_{x \in C_{c,\mathrm{sg}}} \delta(g_x, \boldsymbol{d}_x) \leq g$. In particular, for every $x \in C_{c,\mathrm{sg}}$, the cohomological excess of $\hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)$ along $F^{\delta(h,\boldsymbol{d})} \hat{\mathcal{M}}_{g_x}(\boldsymbol{d}_x)$ is $\leq \delta(h, \boldsymbol{d})$ by assumption. The Lemmas 1.10 and 1.12 then imply that the cohomological excess of $i^!$ at $(0, (o_x)_{x \in C_{c,\mathrm{sg}}})$ is at most $\sum_{x \in C_{c,\mathrm{sg}}} \delta(g_x, \boldsymbol{d}_x) \leq g$. $\square$

4. Proofs of the main theorems

This will proceed by induction. Consider (for $g \geq 1$ and $|\boldsymbol{d}| \geq 2$) the following two properties (which we regard as trivially true for $g = 0$):

A_g: $\mathrm{ce}(\mathcal{C}_h) \leq h - 1$ for all $1 \leq h \leq g$.

B_g: $\mathrm{ce}_{F^k \hat{\mathcal{M}}_h(\boldsymbol{d})} \hat{\mathcal{M}}_h(\boldsymbol{d}) \leq \delta(h, \boldsymbol{d})$ for all k, $h \leq g$ and $\boldsymbol{d}$ with $|\boldsymbol{d}| \geq 2$,

Theorem 0.2 amounts to the validity of A_g for all $g \geq 1$ and so follows with induction on g via the following two assertions.

Assertion 4.1. *If A_{g-1} and B_{g-1} hold, then A_g holds.*

Assertion 4.2. *If A_g and B_{g-1} hold, then B_g holds.*

Proof of Assertion 4.1. We assume that A_{g-1} and B_{g-1} hold and let $\boldsymbol{d}$ be such that $|\boldsymbol{d}| \leq g+1$. We first prove with induction on $m := 2g-2+|\boldsymbol{d}|-|I|$ (but in a number of steps) that $\mathrm{ce}_{F^k \hat{\mathcal{M}}_g(\boldsymbol{d})}(\hat{\mathcal{M}}_g(\boldsymbol{d})) \leq \delta(g, I)$.

Selecting for any effective degree m divisor on $\mathbb{A}^1$ a point in its support defines a degree m covering $\mathbb{A}^1 \times \mathrm{Sym}^{m-1}(\mathbb{A}^1) \to \mathrm{Sym}^m(\mathbb{A}^1)$. Since every $\mathbb{G}_a$-orbit in $\mathbb{A}^1 \times \mathrm{Sym}^{m-1}(\mathbb{A}^1)$ is uniquely represented by a pair whose first coordinate is $0 \in \mathbb{A}^1$, this covering becomes after passage to the $\mathbb{G}_a$-orbit spaces the finite morphism $\mathrm{Sym}^{m-1}(\mathbb{A}^1) \to \mathrm{Sym}_0^m(\mathbb{A}^1)$ which assigns to $\Delta \in \mathrm{Sym}^{m-1}(\mathbb{A}^1)$ the divisor $(0)+\Delta$ modulo translation. We have a filtration $G^\bullet \mathrm{Sym}^{m-1}(\mathbb{A}^1)$ of $\mathrm{Sym}^{m-1}(\mathbb{A}^1)$ by closed subsets with $G^k \mathrm{Sym}^{m-1}(\mathbb{A}^1)$ being the locus of Δ in which 0 appears with multiplicity $\geq k$. This evidently defines an affine stratification. Consider the pull-back diagram

$$\begin{array}{ccc}
\hat{\mathcal{N}}_g(\boldsymbol{d}) & \longrightarrow & \hat{\mathcal{M}}_g(\boldsymbol{d}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Sym}^{m-1}(\mathbb{A}^1) & \longrightarrow & \mathrm{Sym}_0^m(\mathbb{A}^1).
\end{array}$$

Since $\hat{\mathcal{M}}_g(\boldsymbol{d}) \to \mathrm{Sym}_0^m \mathbb{A}^1$ is finite, $G^\bullet \mathrm{Sym}^{m-1}(\mathbb{A}^1)$ pulls back to an affine filtration $G^\bullet \hat{\mathcal{N}}_g(\boldsymbol{d})$. It is given by 'the multiplicity of the singled out point

in the discriminant minus one'. We will focus on the open part $\mathcal{N}_g^\circ(\boldsymbol{d}) \subset \hat{\mathcal{N}}_g(\boldsymbol{d})$ which parametrizes the finite covers $f : C \to \mathbb{P}^1$ for which 0 is in the discriminant, but for which nonetheless the fiber over 0 lies in the smooth part of C. In particular, the multiplicity of 0 in Δ_f is at most g and so $G^g\mathcal{N}_g^\circ(\boldsymbol{d}) = \emptyset$.

Lemma 4.3. *We have* $\mathrm{ce}(\mathcal{N}_g^\circ(\boldsymbol{d})) \le g-1$ *and the discriminant of every member of* $\mathcal{C}_{\mathcal{N}_g^\circ(\boldsymbol{d})}/\mathbb{P}^1_{\mathcal{N}_g^\circ(\boldsymbol{d})}$ *has multiplicity* $< m$ *everywhere.*

Proof. It is clear that $\mathcal{N}_g^\circ(\boldsymbol{d})$ is an open union of strata of $G^\bullet\hat{\mathcal{N}}_g(\boldsymbol{d})$. Hence $G^\bullet\hat{\mathcal{N}}_g(\boldsymbol{d})$ is an affine stratification of depth $\le g-1$ and so Proposition 1.4 implies that $\mathrm{ce}(\mathcal{N}_g^\circ(\boldsymbol{d})) \le g-1$.

If $f : C \to \mathbb{P}^1$ represents a closed point of $\mathcal{N}_g^\circ(\boldsymbol{d})$, then the finite part of its discriminant (which has degree m) has 0 its support with multiplicity between 1 and g. So this discriminant has no point of multiplicity m. □

Proposition 4.4. *We have* $\mathrm{ce}(\mathcal{M}_g(\boldsymbol{d})) \le \delta(g, I) - 1$.

Proof. We prove this with induction on m. Let us denote by $F^\bullet\mathcal{N}_g^\circ(\boldsymbol{d})$ the pull-back of the stratification $F^\bullet\hat{\mathcal{M}}_g(\boldsymbol{d})$. As the map $\mathcal{N}_g^\circ(\boldsymbol{d})) - F^1\mathcal{N}_g^\circ(\boldsymbol{d}) \to \mathcal{M}_g(\boldsymbol{d})$ is finite surjective, it suffices to prove (by Proposition 1.3) that $\mathrm{ce}(\mathcal{N}_g^\circ(\boldsymbol{d}) - F^1\mathcal{N}_g^\circ(\boldsymbol{d})) \le \delta(g, I) - 1$. Since we know that $\mathrm{ce}(\mathcal{N}_g^\circ(\boldsymbol{d})) \le g-1 \le \delta(g, I) - 1$, this will follow (by Proposition 1.7) if we establish that every non-open stratum S of $F^\bullet\mathcal{N}_g^\circ(\boldsymbol{d})$ has the property that $\mathrm{ce}(i_S^!) + \mathrm{ce}(S) \le \delta(g, \boldsymbol{d})$. This is what we will do.

Let S be a non-open stratum of $F^\bullet\mathcal{N}_g^\circ(\boldsymbol{d})$. We invoke Proposition 2.9. If $(g_\alpha, I_\alpha)_{\alpha\in A}$ and $A_+ \subset A$ are as in that proposition, then we have an affine morphism $S \to \prod_{\alpha\in A_+}\mathcal{M}_{g_\alpha, I_\alpha}$. By assumption A_{g-1} and Lemma 1.9 it follows from Proposition 2.9 that

$$\mathrm{ce}(S) \le \textstyle\sum_{\alpha\in A_+}(g(C_\alpha) - 1) = \chi^\psi(C).$$

On the other hand, assumption B_{g-1} and our induction hypothesis imply (via Corollary 2.4) that $\mathrm{ce}(i_S^!) \le \sum_{x\in C}\delta^\psi(C_x)$. According to Lemma 2.8 we have $\sum_x \delta^\psi(C_x) + \chi^\psi(C) \le \delta(g, \boldsymbol{d})$. □

Corollary 4.5. *We have* $\mathrm{ce}_{F^{\delta(g,\boldsymbol{d})}\hat{\mathcal{M}}_g(\boldsymbol{d})}(\hat{\mathcal{M}}_g(\boldsymbol{d})) \le \delta(g, \boldsymbol{d})$.

Proof. We apply Corollary 1.8 to $X := \hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{\delta(g,I)}\hat{\mathcal{M}}_g(\boldsymbol{d})$ with the filtration $X^k := X \cap F^k\hat{\mathcal{M}}_g(\boldsymbol{d})$ and $n = \delta(g, I)$ yields $\mathrm{ce}(\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^k\hat{\mathcal{M}}_g(\boldsymbol{d})) < \max\{\mathrm{ce}(\mathcal{M}_g(\boldsymbol{d})), n-1\} \le \delta(g, I) - 1$. If we combine this with the fact $\hat{\mathcal{M}}_g(\boldsymbol{d})$ is affine, we find the desired estimate $\mathrm{ce}_{F^k\hat{\mathcal{M}}_g(\boldsymbol{d})}(\hat{\mathcal{M}}_g(\boldsymbol{d})) \le \delta(g, I)$ for all k. □

It now follows from Corollary 3.1 that A_g also holds.

Proof of Assertion 4.2. We assume that A_g and B_{g-1} hold. We must prove that for all $\boldsymbol{d}$ with $|\boldsymbol{d}| \geq 2$ we have that $\mathrm{ce}_{F^{\delta(g,\boldsymbol{d})}\hat{\mathcal{M}}_g(\boldsymbol{d})}(\hat{\mathcal{M}}_g(\boldsymbol{d})) \leq \delta(g, I)$. Since $\hat{\mathcal{M}}_g(\boldsymbol{d})$ is affine this is equivalent to: $\mathrm{ce}(\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{\delta(g,\boldsymbol{d})}\hat{\mathcal{M}}_g(\boldsymbol{d})) \leq \delta(g, \boldsymbol{d}) - 1$.

We first observe that $\mathcal{M}_g(\boldsymbol{d})$ is affine over $\mathcal{C}_g$. It then follows from A_g that $\mathrm{ce}(\mathcal{M}_g(\boldsymbol{d})) \leq g-1$. In view of Proposition 1.7 it then suffices to prove that if S is a connected component of $F^k\hat{\mathcal{M}}_g(\boldsymbol{d}) - F^{k+1}\hat{\mathcal{M}}_g(\boldsymbol{d})$ with $1 \leq k \leq \delta(g, I) - 1$, then $\mathrm{ce}(i_S^!) + \mathrm{ce}(S) \leq \delta(g, \boldsymbol{d}) - 1$. It follows from A_{g-1} and Proposition 2.9 that we have $\mathrm{ce}(S) \leq \delta(g, \boldsymbol{d}) - k - 1$. On the other hand B_{g-1} implies that $\mathrm{ce}(i_S^!) \leq k$ and so $\mathrm{ce}(i_S^!) + \mathrm{ce}(S) \leq \delta(g, \boldsymbol{d}) - 1$ as desired.

Remark 4.6. Let for $k = 2, \ldots, g+1$, $W_k\mathcal{C}_g \subset \mathcal{C}_g$ denote the locus which parametrizes pointed curves (C, p) which admit a nonconstant rational function with divisor $\geq k(p)$. If this filtration, introduced by Arbarello in [1], were to define an affine stratification, then Theorem 0.2 would immediately follow from Proposition 1.4, but Arbarello and Mondello have recently shown [2] that this is hardly ever the case. The locus $W_2\mathcal{C}_g$ parametrizes the locus of hyperelliptic curves with a Weierstraß point. This is clearly affine, but we do not know whether for $2 < k \leq g$, $W_k\mathcal{C}_g$ or $W_k\mathcal{C}_g - W_{k+1}\mathcal{C}_g$ has cohomological excess $\leq k-2$.

Proof of Theorem 0.6. For $U = \mathcal{M}_{g,n}$ the assertion is just that of Corollary 0.3. We proceed with induction on the number of strata: suppose this number is > 1 and let $i : S \subset U$ be a stratum that is closed in U so that $\mathrm{ce}(U - S) \leq g - 1 + r(U - S)$ by inductive assumption. Let $(C; p_1, \ldots, p_n)$ represent a closed point of S, denote by $\{\hat{C}_\alpha\}_{\alpha \in A}$ the collection of connected components of the normalization of C, by g_α the genus of $\hat{C}_\alpha$ and by n_α the number of special points on $\hat{C}_\alpha$ (i.e., points that lie over a p_i or over a node). Then we have a finite surjective morphism $\prod_{\alpha\in A}\mathcal{M}_{g_\alpha,n_\alpha} \to S$ and hence by Lemma 1.9 and Proposition 1.3 we have $\mathrm{ce}(S) \leq \sum_\alpha \mathrm{ce}(\mathcal{M}_{g_\alpha,n_\alpha})$. Since $\mathrm{ce}(\mathcal{M}_{g_\alpha,n_\alpha}) \leq (g_\alpha - 1)_+$ by Corollary 0.3, it follows that $\mathrm{ce}(S) \leq \sum_\alpha (g_\alpha - 1)_+ = \chi^\psi(C)$. If C has k nodes of , then k is the codimension of S in U and hence $\mathrm{ce}(i^!) \leq k$. We conclude that $\mathrm{ce}(i^!) + \mathrm{ce}(S) \leq k + \chi^\psi(C)$ and this last expression is just $g - 1 + r(S)$ (by Lemma 2.8). Thus the hypotheses of Proposition 1.7 are fulfilled and we conclude that $\mathrm{ce}(U) \leq g - 1 + r(U)$. □

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